

CONCERNING THE SHAPES OF FINITE-DIMENSIONAL COMPACTA

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ABSTRACT. It is shown that two “tamely” embedded compacta of dimension $\leq k$ lying in E^n ($n \geq 2k + 2$) have the same (Borsuk) shape if and only if their complements are homeomorphic. In particular, two k -dimensional closed submanifolds of E^{2k+2} have the same homotopy type if and only if their complements are homeomorphic.

1. Introduction. We characterize the shapes (in the sense of Borsuk) of compact subsets of a euclidean space in terms of the homeomorphism types of their complements. The characterization is stated in its basic form (Theorem 1.1) and then in a more useful form (Theorem 1.2).

But first some definitions. A closed subset X of a topological space M is a Z_k -set (k an integer ≥ 0) if for every nonempty k -connected open set U in M , $U - X$ is nonempty and k -connected. If M is a metric space and $\epsilon > 0$, we denote the set of points whose distance from X is less than ϵ by $N(X, \epsilon)$. An ϵ -push h of the pair (E^n, X) is a homeomorphism of the n -dimensional euclidean space E^n for which an ϵ -isotopy H of E^n exists satisfying: $H_0 = 1$, $H_1 = h$ and H_t fixes the complement of $N(X, \epsilon)$ for each t in the unit interval I . A closed subset X of E^n is a *strong* Z_k -set ($k \geq 0$) if for each compact subpolyhedron P of E^n having dimension $\leq k + 1$, and each $\epsilon > 0$, there exists an ϵ -push h of (E^n, X) such that $h(X) \cap P = \emptyset$. For the definition of the *shape* of a compactum X , we refer to [4]. As is customary, the notation $\text{Sh}(X) = \text{Sh}(Y)$ means that X and Y have the same shape.

We can now state our main theorems.

Theorem 1.1. *Let X and Y be nonempty compact strong Z_{n-k-2} -sets in E^n ($k \geq 0, n \geq 2k + 2$). Then the following are equivalent:*

- (1) $\text{Sh}(X) = \text{Sh}(Y)$;
- (2) $(E^n/X, \hat{X})$ and $(E^n/Y, \hat{Y})$ are homeomorphic as pairs;
- (3) $(E^n/X, E^n/X - \hat{X}, \hat{X})$ and $(E^n/Y, E^n/Y - \hat{Y}, \hat{Y})$ have the same homotopy type as triples;
- (4) $E^n - X$ and $E^n - Y$ are homeomorphic.

\hat{X} and \hat{Y} denote the quotient points of the quotient spaces E^n/X and E^n/Y . The meaning of “homotopy type of triples” might be guessed, but it is given in §4 before Lemma 4.3.

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The property “strong Z_{n-k-2} ” is not a homotopy property and so it cannot be easily verified in practice. In §2 we review the literature of taming theorems, and show that certain homotopy properties often imply the property strong Z_{n-k-2} . That work enables us to prove

Theorem 1.2. *Let X and Y be nonempty compact subsets of E^n and let either of the following conditions hold:*

(a) $E^n - X$ and $E^n - Y$ are 1-ULC, $n \neq 4$, $n \geq \max(2 \dim X + 2, 2 \dim Y + 2)$, or

(b) X and Y are Z_{n-k-2} -sets, $k \geq 0$, $n \geq 2k + 2$, $n \neq 4$.

Then the following are equivalent:

- (1) $\text{Sh}(X) = \text{Sh}(Y)$;
- (2) $(E^n/X, \hat{X})$ and $(E^n/Y, \hat{Y})$ are homeomorphic as pairs;
- (3) $(E^n/X, E^n/X - \hat{X}, \hat{X})$ and $(E^n/Y, E^n/Y - \hat{Y}, \hat{Y})$ have the same homotopy type as triples;
- (4) $E^n - X$ and $E^n - Y$ are homeomorphic.

As usual 1-ULC means uniformly locally simply connected. Note that there is no explicit dimension restriction in (b) of Theorem 1.2.

Theorem 1.1 suggests the question: Which compacta can be embedded in E^n as strong Z_{n-k-2} -sets? The answer is given by

Proposition 1.3. *Every compactum of dimension $\leq k$ can be embedded in E^n ($n \geq 2k + 1$) as a strong Z_{n-k-2} -set.*

It is time to admit that ours is not the first characterization of shapes of finite-dimensional compacta in terms of homeomorphic complements. The first such theorem (that we know of) is due to Chapman [9] and can be stated as follows:

Let X and Y be compacta of dimension $\leq k$; (i) there exist copies X' (of X) and Y' (of Y) in E^n ($n \geq 2k + 2$) such that if $\text{Sh}(X) = \text{Sh}(Y)$ then $E^n - X$ and $E^n - Y$ are homeomorphic; (ii) there exist copies X' (of X) and Y' (of Y) in E^n ($n \geq 3k + 3$) such that if $E^n - X'$ and $E^n - Y'$ are homeomorphic then $\text{Sh}(X) = \text{Sh}(Y)$.

Theorems 1.1 and 1.2 together with Proposition 1.3 improve Chapman's theorem in two ways: by reducing the unnecessary condition $n \geq 3k + 3$ of (ii) to the more likely $n \geq 2k + 2$; and (more importantly) by making explicit, in elementary terms, which copies X' and Y' in E^n are acceptable.

It is worth pointing out that in the infinite-dimensional case, Chapman has given, in [8], a characterization theorem every bit as strong as our Theorem 1.2, namely:

If X and Y are Z -sets in the Hilbert cube I^∞ , then $\text{Sh}(X) = \text{Sh}(Y)$ if and only if $I^\infty - X$ and $I^\infty - Y$ are homeomorphic.

(“ Z -set” means “ Z_k -set for all $k \geq 0$ ”: a concept first introduced by Anderson in [1] and frequently used in infinite-dimensional topology.) So it would seem

that there are analogies to be found between the point-set topology of euclidean spaces and the Hilbert cube. In another paper [10] we will show just how strong the analogies are, by defining pseudo-boundaries and pseudo-interiors in euclidean spaces, and by showing that many compact subsets of a pseudo-interior (there are several pseudo-interiors) are negligible. However, no knowledge of infinite-dimensional topology is needed for the present paper.

We also draw the reader's attention to Henderson's paper [12]. Many interesting examples are discussed there, and Theorem 1.6 of that paper appears to be related to the $(1) \Rightarrow (3)$ part of our Theorem 1.2 (for the special case $Y = \text{a point}$).

Theorems 1.1 and 1.2 are proved at the end of §4. Proposition 1.3 is simply a weak form of Corollary 3.6.

In §2 we study the relationship between 1-ULC, Z_k and strong Z_k . In §3 we study a special kind of strong Z_k -set in E^n which we call "pseudo-polyhedral": this may have independent interest (see Theorem 3.1) but, for us, it is the means by which we relate the property strong Z_k to shape. Shape itself is only discussed in §4.

This paper can be read independently of [9].

2. Taming theorems. In this section we do the following:

(a) we prove a taming theorem (2.5) for compact strong Z_{n-k-2} -sets in E^n ($n \geq 2k + 2$);

(b) we give conditions under which 1-ULC together with dimension $\leq k$ implies strong Z_{n-k-2} (Theorem 2.2);

(c) we give conditions under which Z_{n-k-2} implies strong Z_{n-k-2} (Theorem 2.4); and hence

(d) we state when 1-ULC sets and Z_{n-k-2} -sets in E^n are tame (Corollary 2.7). In fact, we prove no new taming theorems. We merely recover the theorems of Bryant [5], [6], Štan'ko [16], Bryant-Seebeck [7], Bing-Kister [2] and McMillan [15] (using some of their crucial lemmas). But we believe there is insight to be gained by organizing things in this manner.

A metric space M is k -ULC ($k \geq 0$) if for each $\epsilon > 0$ there exists $\delta > 0$ such that any map of the k -sphere S^k into M whose image has diameter $< \delta$ extends to a map of the $(k + 1)$ -ball B^{k+1} into M whose image has diameter $< \epsilon$. M is said to be ULC^k if M is i -ULC for each $i = 0, \dots, k$.

The relationship between Z_k and ULC^k is given in the following proposition. We give only an outline of the elementary proof.

Proposition 2.1. *Let X be a compact subset of E^n and let k be a nonnegative integer. Then the following are equivalent:*

- (1) X is a Z_k -set;
- (2) $\dim X < n$ and $E^n - X$ is ULC^k ; and
- (3) for each $\epsilon > 0$ and each compact subpolyhedron P of E^n having dimension

$\leq k + 1$, there exists a map $f: P \rightarrow E^n$ such that $f(P) \cap X = \emptyset$ and $d(f(x), x) < \varepsilon$ for each $x \in P$.

Proof. (1) \Rightarrow (2). If U is a nonempty k -connected open subset of E^n , then $U - X$ is nonempty and hence X cannot contain an open subset; i.e., $\dim X < n$. Furthermore, X can be covered by finitely many open sets U_1, \dots, U_m such that $U_i - X$ is nonempty and k -connected for each $i = 1, \dots, m$. By using a typical compactness argument it is easy to see that $E^n - X$ is ULC^k .

(2) \Rightarrow (3). Triangulate P so that the simplexes have small diameter and then, using the fact that $\dim X < n$, define a map f on the 0-skeleton of this triangulation with range missing X . The ULC^k property can then be used to inductively extend f over all the various skeleta.

(3) \Rightarrow (1). Obvious.

Theorem 2.2. *Let X be a nonempty compact subset of E^n having dimension $\leq k$ ($n \neq 4, n - k - 2 \geq 1$) and let $E^n - X$ be 1-ULC. Then X is a strong Z_{n-k-2} -set.*

Proof. Suppose first that $k = 0$. Then it follows from the discussion on p. 707 of [15] that for each $\varepsilon > 0$ there is a finite, disjoint collection of PL n -cells, each of diameter less than ε , whose interiors cover X . But if P is a compact subpolyhedron of E^n having dimension $\leq n - 1$, then there is a small push h of (E^n, X) which moves the union of the PL n -cells, and hence X , off P . This implies that X is a strong Z_{n-2} -set.

Now suppose that $k = 1$. If $\dim X = 0$, then the preceding paragraph applies. We therefore assume that X is 1-dimensional and we let P be a compact subpolyhedron of E^n of dimension $\leq n - 2$. Let $g: X \rightarrow E^n$ be a map such that $g(X)$ is a 1-dimensional compact subpolyhedron of E^n and which moves points a short distance (see [14, Chapter 5]). By general position we may assume that $g(X) \cap P = \emptyset$. Since $n \geq 5$ for this case, we may approximate g by an embedding f such that $f(X) \cap P = \emptyset$. Then applying Lemma 2 of [5] when $n \geq 6$ or Lemma 3 of [6] when $n = 5$, there is a small push h of (E^n, X) such that $h|_X$ is close enough to f to ensure $h(X) \cap P = \emptyset$. Hence X is a strong Z_{n-3} -set.

The remaining case is $k \geq 2$. If this holds, then $n - k - 1 \leq n - 3$ and Lemma 1 of [5] implies directly that X is a strong Z_{n-k-2} -set.

Lemma 2.3. *Let X be a compact strong Z_{n-k-2} -set in E^n . Then $\dim X \leq k$.*

Proof. Let N be a small polyhedral neighborhood of X in E^n which is equipped with a triangulation of small mesh. Let N^k be the body of the k -skeleton and \tilde{N}^{n-k-1} the body of the dual $(n - k - 1)$ -skeleton. Since X is a strong Z_{n-k-2} -set, we can push X off \tilde{N}^{n-k-1} and, since \tilde{N}^{n-k-1} and N^k are "complementary" skeleta, we can then push X arbitrarily close to N^k . This implies $\dim X \leq k$ (see p. 72 of [14]).

Theorem 2.4. *Let X be a nonempty compact Z_{n-k-2} -set in E^n ($k \geq 0, n - k - 2 \geq 1, n \neq 4$). Then X is a strong Z_{n-k-2} -set in E^n .*

Proof. First suppose that $k \geq 2$. By Proposition 2.1, $E^n - X$ is ULC^{n-k-2} . The theorem therefore follows from the proof of Lemma 1 of [5], since $n - k - 1 \leq n - 3$ and the essential property used in that proof is that $E^n - X$ is ULC^{n-k-2} . Now suppose that $k < 2$ and consider $X \times I^2 \subset E^n \times E^2 = E^{n+2}$. It is easily seen that $X \times I^2$ is a $Z_{(n+2)-(k+2)-2}$ -set in E^{n+2} and hence, by the first case we considered, a strong Z_{n-k-2} -set in E^{n+2} . By 2.3, this implies that $\dim(X \times I^2) \leq k + 2$. But then $\dim X \leq k$ by the Remark on p. 34 of [14] and the theorem now follows from 2.2.

Theorem 2.5 (Taming theorem). *Let X and Y be nonempty compact strong Z_{n-k-2} -sets in E^n ($k \geq 0, n \geq 2k + 2$) and let $f: X \rightarrow Y$ be a homeomorphism such that $d(x, f(x)) < \varepsilon$ for each x in X . Then there is an ε -push h of (E^n, X) such that $h|X = f$.*

Theorem 2.5 is proved by combining the techniques of the proof of Theorem 4.4 of [11] with the following.

Lemma 2.6. *Let X be a nonempty compact strong Z_{n-k-2} -set X in E^n ($k \geq 0, n \geq 2k + 2$) and let $f: X \rightarrow E^n$ be an embedding such that $d(x, f(x)) < \varepsilon$ for each x in X . Then for each $\delta > 0$, there is an ε -push h of (E^n, X) such that $d(f(x), h(x)) < \delta$ for each x in X .*

Proof. We shall give only an outline since the proof can be found on pp. 48–49 of [5]. First extend f to a map from E^n to E^n and then push X close to N^k , where N^k is as in the proof of 2.3. But now a taming theorem is known for polyhedra in the trivial range [2] and hence N^k can be pushed close to $f(N^k)$. This last push then drags X along with N^k and thus X is pushed close to $f(X)$.

Corollary 2.7. *Let X and Y be nonempty compact subsets of E^n and let either of the following conditions hold:*

- (1) $E^n - X$ and $E^n - Y$ are 1-ULC, $n \neq 4, n \geq 2 \dim X + 2$; or
- (2) X and Y are Z_{n-k-2} -sets, $k \geq 0, n \geq 2k + 2, n \neq 4$.

Then if $f: X \rightarrow Y$ is a homeomorphism such that $d(x, f(x)) < \varepsilon$ for each x in X , there is an ε -push h of (E^n, X) such that $h|X = f$.

Proof. Case (2) follows directly from Theorems 2.4 and 2.5 except when $(n, k) = (2, 0)$. It is easy to show, however, that a Z_0 -set in E^2 is a strong Z_0 -set. In proving case (1), we may assume $(n - \dim X - 2) \geq 1$: this could only fail to be true when $n = 2$ and $\dim X = 0$, but then $E^n - X$ could not be 1-ULC. Hence case (1) follows from Theorems 2.2 and 2.5. This completes the proof.

We conclude this section with slightly different versions of 2.5 and 2.7 which will be useful in §4.

Theorem 2.8. *Let X and Y be nonempty compact strong Z_{n-k-2} -sets in E^n ($k \geq 0, n \geq 2k + 2$) and let $f: X \rightarrow Y$ be a homeomorphism which is homotopic to 1_X in an open neighborhood U of $X \cup Y$. Then f extends to a homeomorphism h of E^n such that h fixes the complement of U .*

Proof. This follows from 2.5, 2.6 and 2.3. One simply moves along the homotopy in a finite number of stages, using 2.6 and 2.3 to approximate at each stage by an embedding of X whose image is a strong Z_{n-k-2} -set.

Corollary 2.9. *Let X and Y be nonempty compact subsets of E^n and let either of the following conditions hold:*

- (1) $E^n - X$ and $E^n - Y$ are 1-ULC; $n \neq 4, n \geq 2 \dim X + 2$; or
- (2) X and Y are Z_{n-k-2} -sets, $k \geq 0, n \geq 2k + 2, n \neq 4$.

Then if $f: X \rightarrow Y$ is a homeomorphism which is homotopic to 1_X in an open neighborhood U of $X \cup Y$, f extends to a homeomorphism h of E^n such that h fixes the complement of U .

Proof. Combine 2.2, 2.4 and 2.8.

3. Embeddings of compacta in E^n . Let P be a compact subpolyhedron of E^n and let ϵ be a positive real number. An ϵ -regular neighborhood of P in E^n is a regular neighborhood N of P such that for any compact subset Y of $E^n - P$, there is an ϵ -push h of (E^n, P) such that $h(Y) \cap N = \emptyset$. It follows trivially from regular neighborhood theory (e.g., see [13]) that ϵ -regular neighborhoods always exist for any polyhedron P and any $\epsilon > 0$.

Definition 3.0. *Let X be a compact subset of E^n . The statement that X is a pseudo-polyhedral subset of E^n means that there is a pair of sequences $\{P_i\}, \{N_i\}$, $i = 1, 2, \dots$, with the following properties:*

- (1) each P_i is a compact subpolyhedron of E^n ,
- (2) $\dim P_i \leq \dim X$ for each $i = 1, 2, \dots$,
- (3) each N_i is a $(1/i)$ -regular neighborhood of P_i , and
- (4) $X \subset \text{int } N_i \subset N_i \subset N(X, 1/i)$ for each $i = 1, 2, \dots$.

If X is an arbitrary compactum, a pseudo-polyhedral embedding h of X into E^n is an embedding such that $h(X)$ is pseudo-polyhedral.

Note. Any compact subpolyhedron of E^n is a pseudo-polyhedral set and any PL embedding is a pseudo-polyhedral embedding.

Note. The property "pseudo-polyhedral" is more restrictive than the property "standard position" used in [9]. See [17] for an example.

In this section we discuss some properties of pseudo-polyhedral sets for use in §4. One feature of this section which may have independent interest is the following theorem (proved by combining 2.3, 2.8, 3.2, 3.3, and Theorem 1.1 of [11]):

Theorem 3.1. *The strong Z_{n-k-2} -sets in E^n ($k \geq 0, n \geq 2k + 2$) are precisely the images of pseudo-polyhedral sets of dimension $\leq k$ under self-homeomorphisms of E^n .*

Proposition 3.2. *If X is a pseudo-polyhedral set in E^n having dimension $\leq k$ ($n - k - 2 \geq 0$), then X is a strong Z_{n-k-2} -set in E^n .*

Proof. We refer to Definition 3.0. By general position, a compact subpolyhedron of E^n having dimension $\leq n - k - 1$ can be pushed off any one of the polyhedra P_i and so by (3) can be pushed off N_i . This pushes P off X ; hence X can be pushed off P .

Theorem 3.3. *Let X be a compactum having dimension $\leq k$ and let $n \geq 2k + 1$. Then the collection of all pseudo-polyhedral embeddings of X into E^n is a dense G_δ -subset of the collection of all mappings from X into E^n .*

The proof of this theorem will require two lemmas. We note here that the collection of all maps from X into E^n , which we denote by $F[X]$, is assumed to have the uniform topology. Let $I[X]$ denote the embeddings of X into E^n , let $F_p[X]$ denote those f in $F[X]$ for which $f(X)$ is pseudo-polyhedral and $\dim f(X) \leq k$, and let $I_p[X] = F_p[X] \cap I[X]$. If $\varepsilon > 0$, an ε -mapping f of X into E^n is a map such that $\text{diam } f^{-1}(y) < \varepsilon$ for each $y \in E^n$. For $i = 1, 2, \dots$, let G_i denote the set of all $(1/i)$ -mappings in $F[X]$. Then the following lemma is proved on p. 57 of [14].

Lemma 3.4. *Each G_i is a dense open subset of $F[X]$ and hence $I[X] = \bigcap_{i=1}^{\infty} G_i$ is a dense G_δ -subset of $F[X]$.*

For each $i = 1, 2, \dots$, let C_i denote the set of all f in $F[X]$ for which there is a compact subpolyhedron P in E^n , $\dim P \leq k$, and a $(1/i)$ -regular neighborhood N of P such that $f(X) \subset \text{int } N \subset N \subset N(f(X), 1/i)$. Note that $F_p[X] = \bigcap_{i=1}^{\infty} C_i$.

Lemma 3.5. *Each C_i is a dense open subset of $F[X]$.*

Proof. If $f \in F[X]$, then arbitrarily close to f there is a map $g \in F[X]$ whose image is a subpolyhedron of E^n having dimension $\leq k$. Setting $P = g(X)$ and N equal to any $(1/i)$ -regular neighborhood of P , we see that $g \in C_i$. To show C_i is open, let $f \in C_i$ and let P and N be as in the definition of C_i . Then if g is close enough to f (we leave the calculation of ε to the reader), $g(X) \subset \text{int } N \subset N \subset N(g(X), 1/i)$.

Proof of Theorem 3.3. By Lemmas 3.4 and 3.5, $C_i \cap G_i$ is dense and open in $F[X]$ and, by Baire's theorem,

$$\bigcap_{i=1}^{\infty} (C_i \cap G_i) = \left(\bigcap_{i=1}^{\infty} C_i \right) \cap \left(\bigcap_{i=1}^{\infty} G_i \right) = F_p[X] \cap I[X] = I_p[X]$$

is the dense G_δ -subset of $F[X]$.

Corollary 3.6. *Let X be a compactum having dimension $\leq k$ and let $n \geq 2k + 1$. Then every map of X into E^n can be uniformly approximated by an embedding whose image is a strong Z_{n-k-2} -set in E^n .*

4. Shape. We refer the reader to [3] and [4] for definitions concerning the concepts of shape, fundamental sequences, etc. These concepts are defined for compact subsets of a particular absolute retract W (e.g. separable Hilbert space). However Theorem 2.4 of [4] implies that the definitions are really independent of W and of the embeddings in W . This is very useful for us: all our compacta will lie in a euclidean space E^n , and this theorem will allow us to define our fundamental sequences relative to E^n .

In this section, we prove Theorem 1.1 and the necessary supporting lemmas. Theorem 1.2 then follows immediately by combining Theorem 1.1 with the theorems of §2. Lemmas 4.1 and 4.2 give the $(1) \Rightarrow (2)$ part of 1.1; Lemma 4.3 is the main step in proving $(3) \Rightarrow (1)$; and Lemma 4.4 is used in deducing $(4) \Rightarrow (1)$ from $(2) \Rightarrow (3) \Rightarrow (1)$. We end the section with formal proofs of 1.1 and 1.2 in which we combine the lemmas of this section with those of §2 and §3.

The proofs of our next two Lemmas 4.1 and 4.2 are modeled on Henderson's proofs of 4.1 and 1.4 in [12]. Lemma 4.1 is the inductive step in Lemma 4.2.

Lemma 4.1. *Let A and B be nonempty compact strong Z_{n-k-2} -sets in E^n ($k \geq 0, n \geq 2k + 2$) such that $\text{Sh}(A) = \text{Sh}(B)$. Let $\{f_i, A, B\}$ and $\{f'_i, B, A\}$ be fundamental sequences (in E^n) which are homotopy inverse to one another. Let U_0 be a neighborhood of A and let h be a homeomorphism of E^n such that $B \subset h(U_0)$ and $h^{-1} \mid B$ is homotopic to $f'_i \mid B$ in U_0 for almost all i . Then for every neighborhood V_0 such that $B \subset V_0 \subset h(U_0)$, there is a homeomorphism q of E^n such that $q = h$ outside U_0 , $A \subset q^{-1}(V_0)$ and $q \mid A$ is homotopic to $f_i \mid A$ in V_0 for almost all i .*

Proof.⁽²⁾ It is not difficult to choose V an open neighborhood of B in V_0 , U an open neighborhood of A in U_0 and $i_0 \geq 1$ such that for all $i \geq i_0$, $h^{-1} \mid V \simeq f'_i \mid V$ in U_0 , $f_i \mid U \simeq f_{i+1} \mid U$ in V and $f'_i f_i \mid U \simeq 1_U$ in U_0 . By 2.6 and 3.6 we can find an embedding \hat{f} of A into V such that $\hat{f}(A)$ is a strong Z_{n-k-2} -set in E^n and $\hat{f} \simeq f_{i_0} \mid A$ in V . Hence $\hat{f} \simeq f_i \mid A$ when $i \geq i_0$ and we have

$$\begin{aligned} h^{-1} \circ \hat{f} &\simeq h^{-1} \circ (f_i \mid A) \quad \text{in } h^{-1}(V) \\ &= (h^{-1} \mid V) \circ (f_i \mid A) \simeq (f'_i \mid V) \circ (f_i \mid A) \quad \text{in } U_0 \\ &= (f'_i f_i) \mid A \simeq 1_A \quad \text{in } U_0. \end{aligned}$$

⁽²⁾ Here and throughout §4, the notation " $a \simeq b$ in T " means that the maps a and b are homotopic in the subspace T .

Hence $\hat{f} \simeq h \mid A$ in $h(U_0)$. By Theorem 2.8 there is a homeomorphism r of E^n such that $r = 1$ outside $h(U_0)$ and $\hat{f} = rh \mid A$. Let $q = r \circ h$. Then $q \mid A = \hat{f}$ and so q has the required properties.

Lemma 4.2. *Let X and Y be nonempty compact strong Z_{n-k-2} -sets in E^n ($k \geq 0, n \geq 2k + 2$) such that $\text{Sh}(X) = \text{Sh}(Y)$. Then there is a homeomorphism of pairs between $(E^n/X, \hat{X})$ and $(E^n/Y, \hat{Y})$.*

Proof. Let $\{T_i\}$ and $\{W_i\}$ be basic systems of neighborhoods for X and Y respectively. Let $\{g_i, X, Y\}$ and $\{g'_i, Y, X\}$ be homotopy inverse fundamental sequences (in E^n). Apply 4.1 with $A = X, B = Y, f_i = g_i, f'_i = g'_i, U_0 = E^n, h = 1$ and $V_0 = W_1$ to obtain $q = p_1$, a homeomorphism of E^n such that $X \subset p_1^{-1}(W_1)$ and $p_1 \mid X \simeq g_i \mid X$ in W_1 for almost all i . Our final homeomorphism will equal p_1 on $E^n - p_1^{-1}(W_1)$. Let $S_2 = T_2 \cap p_1^{-1}(W_1)$. Next, apply 4.1 with $A = Y, B = X, f_i = g'_i, f'_i = g_i, U_0 = W_1, h = p_1^{-1}$ and $V_0 = S_2$ to obtain $q = p_2^{-1}$, a homeomorphism of E^n such that $p_2 = p_1$ on $E^n - p_1^{-1}(W_1)$, $Y \subset p_2(S_2)$ and $p_2^{-1} \mid Y \simeq g'_i \mid Y$ in S_2 for almost all i . Our final homeomorphism will equal p_2 on $E^n - S_2$. Let $S_3 = W_3 \cap p_2(S_2)$. Next, apply 4.1 with $A = X, B = Y, f_i = g_i, f'_i = g'_i, U_0 = S_2, h = p_2$ and $V_0 = S_3$ to obtain $q = p_3$ such that $p_3 = p_2$ on $E^n - S_2$, $X \subset p_3^{-1}(S_3)$ and $p_3 \mid X \simeq g_i \mid X$ in S_3 for almost all i . Continue in this way, defining a homeomorphism $p : E^n - X \rightarrow E^n - Y$ by $p = p_{2i}$ on $E^n - S_{2i}$. Since $S_{2i} \subset T_i$, $\{S_{2i}\}$ is a basic system of neighborhoods of X , and since $p_{2i}(S_{2i}) \subset S_{2i-1} \subset W_{2i-1}$, $\{p_{2i}(S_{2i})\}$ is a basic system of neighborhoods of Y . Thus p extends to a homeomorphism of pairs between $(E^n/X, \hat{X})$ and $(E^n/Y, \hat{Y})$.

Before proving Lemma 4.3, we give some definitions. For $i = 0$ or 1 let A_i be a space with a base point $a_i \in A_i$. Letting $j = 1$ when $i = 0$ and $j = 0$ when $i = 1$, we say $(A_0, A_0 - a_0, a_0)$ and $(A_1, A_1 - a_1, a_1)$ have the same *homotopy type as triples* if there are maps $f_i : A_i \rightarrow A_j$ and homotopies $F_i : A_i \times I \rightarrow A_i$ such that $F_i(\cdot, 0) = f_j \circ f_i, F_i(\cdot, 1) = 1_{A_i}, F_i(\{a_i\} \times I) = \{a_i\}$ and $F_i((A_i - \{a_i\}) \times I) \subset A_i - \{a_i\}$. f_i is a *triple homotopy equivalence* and f_j is its *triple homotopy inverse*.

Lemma 4.3. *Let X and Y be disjoint nonempty compact pseudo-polyhedral subsets of $E^{2k+1} \subset E^n$ ($n \geq 2k + 2$) having dimension $\leq k$. Let $(E^n/X, E^n/X - \hat{X}, \hat{X})$ and $(E^n/Y, E^n/Y - \hat{Y}, \hat{Y})$ have the same homotopy type as triples. Then $\text{Sh}(X) = \text{Sh}(Y)$.*

Proof. Since X is pseudo-polyhedral in E^{2k+1} , there exist polyhedra $\{P_i\}$ and regular neighborhoods $\{N_i\}$ as in Definition 3.0, and we may assume (by choosing subsequences) that $N_{i+1} \subset \text{int } N_i$ for all i . If $r_i : N_i \rightarrow P_i$ is (the final stage of) a PL strong deformation retraction, it follows that for all $j > i$, $r_i \mid N_j \simeq r_j$ in $\text{int } N_i$. By taking the cartesian product of each N_i with a small PL ball, we can enlarge N_i to be a regular neighborhood of P_i in E^n , and r_i can be extended to this enlarged neighborhood by the projection map of E^n onto E^{2k+1} . Abusing notation we call the new neighborhoods and retractions N_i and r_i , and we observe

$r_i | N_j \simeq r_j$ in $(\text{int } N_i \cap E^{2k+1})$ for all $j > i$. Similarly, we let $\{P'_i\}$, $\{N'_i\}$ and $\{r'_i\}$ be the polyhedra, (enlarged) regular neighborhoods and (extended) retractions corresponding to Y . Since $X \cap Y = \emptyset$ we may assume $N_1 \cap N'_1 = \emptyset$.

When $0 \leq t \leq 1$, define $F_t : E^{2k+1} \rightarrow E^n$ by $F_t(x_1, \dots, x_{2k+1}, 0, \dots, 0) = (x_1, \dots, x_{2k+1}, t, 0, \dots, 0)$. Clearly we may assume $F_{1/i}(N_i \cap E^{2k+1}) \subset N_i$ and $F_{1/i}(N'_i \cap E^{2k+1}) \subset N'_i$ for each i .

Let $\hat{h} : E^n/X \rightarrow E^n/Y$ be the triple homotopy equivalence given in the hypothesis, and let \hat{h}' be its triple homotopy inverse. Let $h : E^n - X \rightarrow E^n - Y$ and $h' : E^n - Y \rightarrow E^n - X$ be the obvious induced maps. Because h and h' come from \hat{h} and \hat{h}' , we can choose $n(i) \geq i$ and $n'(i) \geq i$ to satisfy $h' \circ h | (N_{n(i)} - X) \simeq 1$ in $N_i - X$, $h \circ h' | (N'_{n'(i)} - Y) \simeq 1$ in $N'_i - Y$, $h(N_{n(i)} - X) \subset \text{int } N'_i$ and $h'(N'_{n'(i)} - Y) \subset \text{int } N_i$. We may assume $n(j) \geq n(i)$ and $n'(j) \geq n'(i)$ whenever $j > i$.

Now we define our fundamental sequences. Let $f_1 : E^n \rightarrow E^n$ be any continuous map which agrees with $h \circ F_{1/n(1)} \circ r_{n(1)}$ on $N_{n(1)}$; f_1 maps $N_{n(1)}$ into $\text{int } N'_1 - Y$. Next we define $f_2 : E^n \rightarrow E^n$ in three stages as follows: (i) f_2 agrees with $h \circ F_{1/n(2)} \circ r_{n(2)}$ on $N_{n(2)}$ (and therefore f_2 maps $N_{n(2)}$ into $\text{int } N'_2 - Y$); (ii) since $f_2 | N_{n(2)} \simeq f_1 | N_{n(2)}$ in $\text{int } N'_1 - Y$ [proof: $r_{n(1)} | N_{n(2)} \simeq r_{n(2)}$ in $(\text{int } N_{n(1)} \cap E^{2k+1})$, hence $F_{1/n(2)} \circ (r_{n(1)} | N_{n(2)}) \simeq F_{1/n(2)} \circ r_{n(2)}$ in $(\text{int } N_{n(1)} - E^{2k+1})$, hence $F_{1/n(1)} \circ r_{n(1)} | N_{n(2)} \simeq F_{1/n(2)} \circ r_{n(2)}$ in $(\text{int } N_{n(1)} - E^{2k+1})$, and the claim follows by applying h to each side], the Borsuk extension theorem allows us to extend f_2 to $N_{n(1)}$ so that $f_2 | N_{n(1)} \simeq f_1 | N_{n(1)}$ in $\text{int } N'_1 - Y$; (iii) f_2 is extended to the rest of E^n in an arbitrary fashion. By repeating this procedure, we can define inductively $f_i : E^n \rightarrow E^n$ to agree with $h \circ F_{1/n(i)} \circ r_{n(i)}$ on $N_{n(i)}$ (it maps $N_{n(i)}$ into $\text{int } N'_i - Y$) and such that if $1 \leq j < i$, $f_i | N_{n(j)} \simeq f_j | N_{n(j)}$ in $(\text{int } N'_j - Y)$. Similarly, interchanging primed and unprimed symbols, we define maps $f'_i : E^n \rightarrow E^n$. The definitions make $\underline{f} = \{f_i, X, Y\}$ and $\underline{f}' = \{f'_i, Y, X\}$ fundamental sequences.

To prove $\underline{f}' \circ \underline{f}$ is (fundamentally) homotopic to the identity sequence of X , we need only show that for any i , and for any $j \geq n(n'(i))$, $f'_j \circ f_j | N_j \simeq 1_{N_j}$ in $\text{int } N_i$. Let $Q_j = F_{1/j} r_j(N_j)$. Then Q_j is a subpolyhedron of N_j having dimension $\leq k$, and $Q_j \cap X = \emptyset$. $h(Q_j)$ lies in $(\text{int } N'_{n'(i)} - Y)$. Let $e_j : Q_j \rightarrow \text{int } N'_{n'(i)} - Y$ be a PL map homotopic in $(\text{int } N'_{n'(i)} - Y)$ to $h | Q_j$. Clearly e_j is PL homotopic in $\text{int } N'_{n'(i)}$ to $F_{1/n'(i)} \circ r'_{n'(i)} \circ e_j$ (which also maps Q_j into $\text{int } N'_{n'(i)} - Y$). Since the image of this PL homotopy has dimension $\leq k + 1 \leq n - k - 1$, Proposition 3.2 allows us to move the homotopy off Y and to assert $e_j \simeq F_{1/n'(i)} \circ r'_{n'(i)} \circ e_j$ in $(\text{int } N'_{n'(i)} - Y)$. Hence $h | Q_j \simeq F_{1/n'(i)} \circ r'_{n'(i)} \circ h | Q_j$ in $(\text{int } N'_{n'(i)} - Y)$. Applying h' to each side, replacing Q_j by $F_{1/j} r_j(N_j)$, and using the homotopies $F_{1/j} \circ r_j \simeq 1_{N_j}$ in N_j , and $h' \circ h | N_j \simeq 1_{N_j}$ in $\text{int } N_i$, we find $1_{N_j} \simeq f'_{n'(i)} \circ f_j | N_j$ in $\text{int } N_i$. Since $f'_{n'(i)} | N'_{n'(i)} \simeq f'_j | N'_{n'(i)}$ in $\text{int } N_i$, we conclude $f'_j \circ f_j | N_j \simeq 1_{N_j}$ in $\text{int } N_i$ as was to be shown. Similarly $\underline{f} \circ \underline{f}'$ is fundamentally homotopic to the identity sequence of Y .

Lemma 4.4. *Let X and Y be compacta, let x be a discrete point of $X = X' \cup \{x\}$, let y be a discrete point of $Y = Y' \cup \{y\}$, and let $\text{Sh}(X) = \text{Sh}(Y)$. Then $\text{Sh}(X') = \text{Sh}(Y')$.*

Proof. Let f and g be mutually homotopy inverse fundamental sequences between X and Y . By Lemma 10.1 of [3] there is a bijection between the components of X and the components of Y such that corresponding components have the same shape: let X_1 correspond to $\{y\}$ and let Y_1 correspond to $\{x\}$. Then $\text{Sh}(X_1) = \text{Sh}(Y_1) = \text{Sh}(\text{point})$. The proof of Lemma 10.1 of [3] shows more: namely, that the fundamental equivalences between corresponding components are all obtained by suitably restricting f and g . This is easily seen to imply that $X - X_1$ and $Y - Y_1$ are compact. Therefore, by suitably restricting f and g we have $\text{Sh}(X - (X_1 \cup \{x\})) = \text{Sh}(Y - (Y_1 \cup \{y\}))$. If $X_1 = \{x\}$ then $Y_1 = \{y\}$ and we are done. If $X_1 \neq \{x\}$ then X' [resp. Y'] is the union of the two disjoint compacta $X - (X_1 \cup \{x\})$ and X_1 [resp. $Y - (Y_1 \cup \{y\})$ and Y_1]. Clearly this implies $\text{Sh}(X') = \text{Sh}(Y')$.

Proof of Theorem 1.1. (1) \Rightarrow (2) is Lemma 4.2. (2) \Rightarrow (3) is trivial. (2) \Rightarrow (4) is trivial.

Proof of (3) \Rightarrow (1). By 2.3, X and Y have dimension $\leq k$. Hence Theorem 3.3 allows us to find disjoint pseudo-polyhedral sets X' and Y' lying in $E^{2k+1} \subset E^n$, such that X is homeomorphic to X' and Y is homeomorphic to Y' . By Proposition 3.2, X' and Y' are strong Z_{n-k-2} -sets in E^n . By Theorem 2.8, there is a homeomorphism of E^n taking X onto X' , and another taking Y onto Y' . Hence $(E^n/X', E^n/X' - \hat{X}', \hat{X}')$ and $(E^n/Y', E^n/Y' - \hat{Y}', \hat{Y}')$ have the same homotopy type as triples. Lemma 4.3 implies $\text{Sh}(X') = \text{Sh}(Y')$ and hence $\text{Sh}(X) = \text{Sh}(Y)$.

Proof of (4) \Rightarrow (1). Let $S = E^n \cup \{z\}$ be the one point compactification of E^n , and let $z' \in S - (X \cup Y \cup \{z\})$. By hypothesis there is a homeomorphism $h : S - (X \cup \{z\}) \rightarrow S - (Y \cup \{z\})$ and we can arrange that $h(z') = z'$. Let $h' : S - \{z\} \rightarrow E^n$ be a homeomorphism, let $X' = h'(X \cup \{z\})$ and let $Y' = h'(Y \cup \{z\})$. Then $h' \circ h \circ (h')^{-1}$ induces a pair-homeomorphism between $(E^n/X', \hat{X}')$ and $(E^n/Y', \hat{Y}')$. Since (2) \Rightarrow (3) \Rightarrow (1), we conclude $\text{Sh}(X') = \text{Sh}(Y')$. Lemma 4.4 then implies $\text{Sh}(X) = \text{Sh}(Y)$.

Proof of Theorem 1.2. In case of hypothesis (a), let $k = \max(\dim X, \dim Y)$. We may assume $n - k - 2 \geq 1$: this only fails if $n = 2$ and $k = 0$, but then $E^n - X$ or $E^n - Y$ fails to be 1 - ULC. Now combine 2.2 with 1.1. In case of hypothesis (b), combine 2.4 with 1.1.

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